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Visually understanding the arithmetic progression formula: a didactic experience through figurate numbers, geometry, and the History of Mathematics*

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**Free translation provided by the author(s) of the original published article, entitled "Entendendo a fórmula da progressão aritmética visualmente: uma experiência didática através de números figurados, geometria e História da Matemática ".*

Augusto Mendes Duarte[1]

Abstract: Arithmetic Progression (AP) is a mathematical topic pertinent to High School education that is typically taught and demonstrated algebraically, often reducing the content to formula memorization. The only commonly made historical consideration about this topic is the anecdote of Gauss, recounting his childhood genius - an elitist narrative that contrasts with the trajectory of the average student. With such issues in mind, a didactic project was developed that articulates the History of Mathematics, manipulative materials, and problem-solving to teach AP in a contextualized manner. The goal of this article is to describe this approach so that other Math teachers can also benefit from it. Given that sequences were extensively studied by the Pythagoreans in Antiquity using the visual resource of figurate numbers, such a context was chosen to situate the study of AP. Through investigative activities with colored chips, it was demonstrated in the classroom how the formulas of AP can be visually deduced from triangular numbers - a type of figurate number of great historical importance. Finally, using this manipulative material, it was pictorially demonstrated why the formula for the sum of AP terms is identical to the formula for the area of a trapezoid. Through a final evaluative activity, students demonstrated having developed a pictorial and algebraic understanding of APs, successfully solving complex problems. The didactic experience proved to be a success, considering the students' positive perception and the articulation of diverse contents developed and executed successfully.

Keywords: Arithmetic Progression; History of Math; Manipulative Materials; Problem-solving.

Entendendo a fórmula da progressão aritmética visualmente: uma experiência didática através de números figurados, geometria e História da Matemática

Resumo: Progressão Aritmética (PA) é um conteúdo matemático pertinente ao Ensino Médio que é normalmente ensinado e demonstrado algebricamente, resumindo-se à memorização de fórmulas. A única consideração histórica comumente feita acerca deste tema é a anedota de Gauss, que relata sua genialidade infantil - um relato elitista que contrasta com a trajetória do aluno comum. Tendo tais problemáticas em vista, desenvolveu-se um projeto didático que articula História da Matemática, materiais manipulativos e resolução de problemas para ensinar PA de maneira contextualizada. Este relato tem o objetivo de descrever tal abordagem para que a comunidade de educadores matemáticos possa dela usufruir. Dado que sequências foram estudadas profusamente pelos Pitagóricos na Antiguidade utilizando-se do recurso visual dos números figurados, tal contexto foi escolhido para situar o estudo da PA. Através de atividades investigativas com fichas coloridas, demonstrou-se em sala de aula como as fórmulas da PA podem ser deduzidas visualmente dos números triangulares - um tipo de número figurado de grande importância histórica. Por fim, utilizando-se deste material manipulativo, demonstrou-se pictoricamente o porquê da fórmula da soma dos termos da PA ser idêntica à fórmula da área do trapézio. Através de uma atividade avaliativa final, os alunos demonstraram ter desenvolvido um raciocínio pictórico e algébrico sobre PAs, conseguindo resolver problemas complexos apropriadamente. A experiência didática mostrou-se um sucesso considerando também a boa percepção

[1] Licentiate Degree in Mathematics, Universidade Federal do ABC (UFABC). E-mail: augusto-md@hotmail.com- OCID: https://orcid.org/0000-0003-4952-6073.

revista PARANAENSE DE EDUCAÇÃO MATEMÁTICA

dos alunos e as articulações de conteúdos diversos desenvolvidas e executadas com êxito.

Palavras-chave: Progressão Aritmética; História da Matemática; Materiais Manipulativos; Resolução de Problemas.

1 Introduction

Arithmetic progression is a content typically addressed in high school, following up on topics about sequences previously studied in middle years, as guided by the Base Nacional Comum Curricular - BNCC (Brasil, 2018), the central document guiding Brazilian basic education.

> (EM13MAT507) Identifying and associating numerical sequences (AP) with linear functions of discrete domains for property analysis, including deducing some formulas and problem-solving (BRASIL, p.533, 2018, translated by the author).

However, it is common for such content to be presented solely as a formula set, without context or a deep understanding of other relevant aspects.

When contextualization is offered, it is usually done through the famous anecdote of Gauss, which praises his genius and positions him above other children and adults, reflecting an elitist view of mathematical knowledge. Subsequently, the content is presented in an abstract and algebraic manner, often lacking visuality and context.

This article aims to discuss a didactic project carried out with first-year high school classes. The scope of this experience was to articulate different teaching approaches, aiming to provide better contextualization and visuality for such important content. The results of the project were assessed considering its applicability and effectiveness.

2 Theoretical Basis

In BNCC, it is argued that:

> In addition to various teaching resources and materials such as gridded paper, abacuses, games, calculators, spreadsheets, and dynamic geometry softwares, it is important to include the history of mathematics as a resource that can spark interest and provide a meaningful context for learning and teaching mathematics. However, these resources and materials need to be integrated into situations that encourage reflection, contributing to the systematization and formalization of mathematical concepts (BRASIL, 2018, translated by the author).

Through these general recommendations, the importance of the recurrent use of diverse approaches in mathematics teaching is briefly illustrated. The teaching methods chosen for this didactic project were: (i) Manipulative Materials; (ii) History of Mathematics; (iii) Problem Solving. We can further justify the choice of these approaches beyond the BNCC, as they have been studied for decades in extensive research on teaching.

Since Brazilian research such as the one by Miguel (1997), the History of Mathematics has been pointed out as capable of demystifying the discipline and making it more accessible to students. At the same time, it provides a broader social context, underpinning the importance of mathematical knowledge.

Regarding the specific content addressed in this study, it is common for teachers to present the anecdote of Gauss to introduce the content historically: it is said that he, as a child, was instructed to sum all numbers from 1 to 100, as a task to take time. However, he reportedly solved it in just a few minutes, from noticing that the sum of 1 and 100, 2 and 99, and so on, are all equal to 101, applying the basic idea behind the arithmetic progression formula.

Such history, however, does not align with the authors' suggestions regarding the applications of the History of Mathematics in the classroom: History would serve, as proposed by Saito (2013) and Miguel (1997), as a device capable of recreating the exhaustive efforts of generations of mathematicians, highlighting their errors, difficulties, and conflicts. This type of contextualization would help students identify with the sometimes arduous process of mathematical knowledge, envisioning how errors and difficulties have always been common among mathematicians of all eras. This is precisely the opposite of what Gauss's anecdote conveys, as it suggests the elitist view that mathematics can only be done by geniuses - by gifted children who invent new ways of calculating, emphasizing their difference from our students, mere mortals.

The use of history is of paramount importance for the present article, as it is used to provide a broader contextualization of the study of sequences, breaking the aforementioned traditional trend of presenting a formula-ridden arithmetic progression. With this in sight, it is intended to allow students to glimpse different ways of understanding sequences, tacitly understanding different representations used throughout history.

Another approach supported by recent research, such as that of Brown *et al.* (2013), is Problem Solving. As suggested, perceiving content in different contexts is beneficial for learning, with true learning occurring when students are challenged to recall and apply content in diverse contexts. Transforming the abstract material of the classroom into real problems with tangible objects was one objective of this project.

Problems play a crucial role in the development of students' confidence and self-esteem, as argued by Onuchic and Allevato (2011), since facing challenges stimulate a deeper understanding of the content. Brown (2014) emphasizes that the acquisition of knowledge should require effort because whatever is laborious for the mind to build and organize should be more easily remembered in the future. Therefore, mathematical problem solving, combined with other mentioned approaches, supposedly promotes more effective learning.

Also according to Brown *et al.* (2014), the teaching practice of mathematics often contradicts current advances in the science of learning. It is common for each concept to be fragmented and presented in isolation, being repeated until it is "memorized," at which point the class moves to the next topic. However, current scientific evidence suggests that a more effective approach would be to teach related concepts simultaneously, as context and a variety of stimuli favor information retention - an approach opposite to the traditional one.

This perspective constitutes a strong justification for contextualized teaching, in which knowledge is integrated into real-life application situations, thus increasing its retention in the student's mind. Furthermore, the historical contextualization of mathematics not only facilitates the learning of this discipline but also enriches the understanding of history itself, aligning with the guidelines of the BNCC.

The choice to use concrete materials, in turn, followed the guidelines proposed by Lorenzato (2006), who advocates the playful and stimulating benefits of these resources. Especially concerning high school students, a stage at which such materials are commonly discarded as didactic possibilities, their use intended to rescue concrete thinking in its ludic capacities.

The use of concrete materials for study is also highlighted by Roque (2012) and Heath (1921) as a common practice over many centuries among mathematicians. In this way, the described activity also provided students with an immersion in the ancient practice of mathematics, allowing them to experience firsthand the relationship between formal knowledge and its practical applications.

We can also compare such an approach with traditional theories of learning. Ausubel (2003), for example, emphasized the importance of activating prior knowledge to connect new learning. He argued that meaningful learning depends on connection with pre-existing knowledge, an idea corroborated by the research led by Brown (2014). In this context, this project utilized a variety of students' prior knowledge to underpin the study, incorporating concepts about sequences previously studied in elementary school.

Finally, it is worth highlighting some prior studies that have sought to articulate the history of figurate numbers with arithmetic progression. Chiconello (2013) presents the latter as useful for the study of sequences, while Yassunaga and Bortoli (2024) and Alves and Barros

(2019) explore figurate numbers through the digital environment of GeoGebra. Masseno and Pereira (2020), in turn, propose an activity aimed at identifying the pattern in the terms of triangular numbers. These approaches, however, either reference history only briefly or do not address it at all. Moreover, they never arrive at the general formula for arithmetic progression. Therefore, these prior works neither articulate the various approaches developed in the present project nor establish a connection between figurate numbers and arithmetic progression, which is a central topic in secondary education.

3 Didactic Proposal

The project consisted of a series of activities implemented in a first-year high school class at a private institution in São Paulo. The class comprised 22 students, with an average age of approximately 16 years. These activities were spread out over four 50-minute classes throughout one week. Table 1 illustrates this distribution.

Table 1: Didactic proposal sequence.

Class	Goal	Methodology and Resources
1. Introduction: History of Math	Discussing what students recall about the History of Mathematics, specifically about the Pythagoreans, describing their historical relevance in Ancient Greece	Interactive exposition
2. Triangular, square and oblong numbers	Visually understanding triangular, square, and oblong numbers, exploring how such sequences operate	Manipulative material: 500 colored chips
3. Visualizing the triangular number formula	Deriving formulas for triangular numbers using figurate numbers	Manipulative material: 500 colored chips
4. Visualizing the AP formula	Algebraically demonstrating equivalences between formulas until reaching the simplest one, ultimately explaining it visually	Solving and group investigation
5. Evaluative problems	Assessing students' understanding through application problems of APs.	Group solving

Source: developed by the author (2024).

The itinerary was chosen according to the necessary concepts to derive the formula for the sum of an arithmetic progression using figurate numbers. To achieve this ultimate goal, it is necessary to understand triangular numbers; to deduce the formula for these, it is indispensable to understand square or oblong numbers. History permeates all these concepts, as they were extensively studied by the Pythagoreans. A historical introduction was offered in the first class, while in subsequent Mathematics classes, historical connections were interspersed with mathematical content, avoiding the presentation of History as a mere addendum by contextualizing it in each situation.

3.1 Introduction: History of Math

Interactive slides were used to discuss with students the historical content, illustrating what was being discussed. Regarding the Pythagoreans, a historical itinerary based on another article from this journal (Mendes and Yamamoto, 2022) was used, in which the authors describe in detail the history of this doctrine, as taught in a ninth-grade class about the Pythagorean Theorem.

Given that such content is available in another publication, this article will only provide a brief summary of this historical contribution, adding new content relevant to sequences. Interested readers are suggested to consult the mentioned article, as the details provided to students are important for proper contextualization of the activity in the classroom. Here, brief historical information will be provided to help the reader understand how Mathematics is situated in the Pythagorean context.

According to Roque (2012), there existed an important religious and philosophical sect in Greece known as "The Pythagoreans". It is alleged that this group was founded by a man named Pythagoras, whose life is shrouded in mystery. Today, it cannot be certain whether he even existed, since contemporary reports all refer to the group, not to the leader.

As explained by Mendes and Yamamoto (2022), Pythagorean studies encompassed much more than the mathematics for which they are known: they developed philosophies, religions, and lifestyles distinct from those of the time, having a good deal of intellectual independence.

This in-depth contextualization was sought not only as a "footnote" or "curiosity", but rather as a serious reflection on the historical functions of such philosophers. To do so, a conversation was initiated with the students in which they exercised their prior knowledge of Ancient History.

It is noteworthy, for example, how Ancient Greece is an important topic for the middle years, as emphasized by the BNCC: "(EF06HI10) Explain the formation of Ancient Greece, with emphasis on the formation of the polis and on political, social, and cultural transformations" (Brasil, 2018, translated by the author).

However, this skill is said to belong only to the Sixth Year of middle years. There is no mention of Ancient Greece afterward, neither in middle years nor in high school. This fact makes it more strategic to review and debate this content with high school students, discussing such knowledge in light of all the other historical knowledge acquired over the years.

Thus, conversations were held with the students about what they remembered from

Ancient Greece and how it fit into the flow of history. What was the politics of that time? Was it a democracy like Brazil today? Was it a united country or a fragmented government? What freedom would a group like the Pythagoreans have to have ideas different from the common ones?

This discussion was grounded in the accounts of Roque (2012), focused on Mathematics, as well as those of Lefèvre (2013). It sought to emphasize to the students how mathematicians were also citizens, many of whom were involved in areas now labeled as humanities: they were politicians, philosophers, merchants, etc. The Pythagoreans are a great example of this polymathy and therefore serve as the central axis of connection between Mathematics and society.

By remaining cautious to historically contextualize each concept, a fundamental mathematical representation for the Pythagoreans was introduced: the figurate numbers. According to Figure 1, these are pebbles placed in sequence to represent a number, concept, or to study properties. According to Roque (2012), such "pebble arithmetic" was the core of Pythagorean mathematical study, not the geometry for which the famous Pythagorean Theorem is known.

Figure 1: Figurate number examples.

Source: developed by the author (2024).

The first and second figurate numbers from the previous figure are of special importance to the Pythagoreans, being equally fundamental for the study of sequences. They are examples of the class of triangular and square numbers, respectively. Such numbers held philosophical, religious, and cultural significance for the Pythagoreans, with the triangular number of 10 pebbles even being used as a symbol for the Pythagorean school. Once sufficient historical contextualization was added to those provided by Mendes and Yamamoto (2022), the students would start working with figurate numbers in a concrete manner.

3.2 Triangular, square, and oblong numbers

Questions related to triangular numbers were projected on the board, which students investigated in groups of four or five. The following image was provided as guidance.

Figure 2: The first three triangular numbers.

Source: developed by the author (2024).

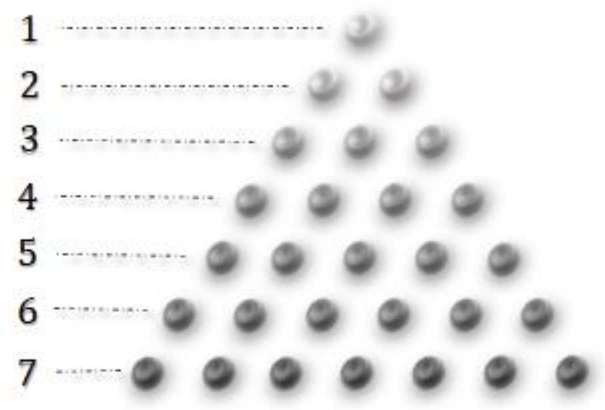
The questions guided the students to assemble the sequence of the first five triangular numbers using colored chips. The physical material allowed them to freely alter the shapes constructed, concretizing their thoughts more efficiently. The chips were used with great enthusiasm since it had been many years since the students had a Math class with special materials.

The problems offered were to be answered in the notebook, along with a sketch of the figures, when possible. They were:

1. In the constructed sequence, how much does it increase from one triangle to the next?
2. So, what is the pattern that governs the increase between triangular numbers?
3. How many more chips will the sixth triangle have compared to the fifth?
4. How many more will the 19th triangular number have compared to the 18th?

Only one group failed to solve the questions within the provided 15 minutes. The results obtained were then discussed in a plenary session - students from each group explained their reasoning in their own words, complementing each other. The teacher then summarized on the board the reached conclusion. "To obtain the second triangular number, how many pebbles do I add to the first? And to obtain the third? And the fourth?" The students responded in unison. A diagram was created as represented in the following figure.

Figure 3: Successive assembly of a triangular number, with each row added below.



Source: developed by the author (2024).

It was deduced that the sequence adds, to each term, the next natural number. Next, the class discussed other functions and interpretations of such numbers, as mentioned by Mendes and Yamamoto (2022): Aristotle, for example, interpreted the previously mentioned fourth triangular number as a representation of the three dimensions. Therefore, such figures carried metaphysical representations beyond the strictly mathematical.

Efforts were made to contextualize, including figurate numbers that can be seen in everyday life. "Fruits at the market", "windows in a building", and "chocolate bars" were examples provided by the students. Connecting with other disciplines, we briefly discussed how figurate numbers are present in a molecule model that was in the room.

Figure 5: Example of real-life figurate numbers.

Source: https://pt.vecteezy.com/foto/1990287-laranja-fruta-pilha.

Again equipped with the chips, the next step was to study the sequence of square numbers through analogous problems. As illustrated in Figure 5, the first three terms of the sequence of square numbers were provided to the students, having their investigation guided by problem questions.

Figure 5: First three terms of the square number sequence.

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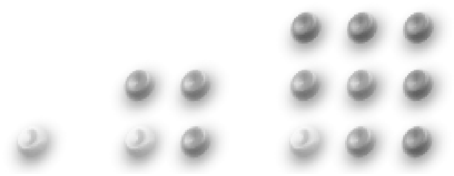


Source: developed by the author (2024).

1. How much does it increase from one square to the next?
2. Then, what is the pattern that governs the increase between square numbers? What is the name of the set of numbers composed of the difference between consecutive squares?
3. How many more chips will the seventh square have compared to the sixth?
4. How many more will the 28th square number have compared to the 27th?

Once they had already solved for triangles, the groups had no difficulty in solving most of the questions for square numbers. Noticing and naming the set discovered in question 2 (the odd numbers), however, required a bit more creativity, as it is not as immediate as in the case of the triangles. One of the groups needed a hint since they couldn't verbalize what they had realized.

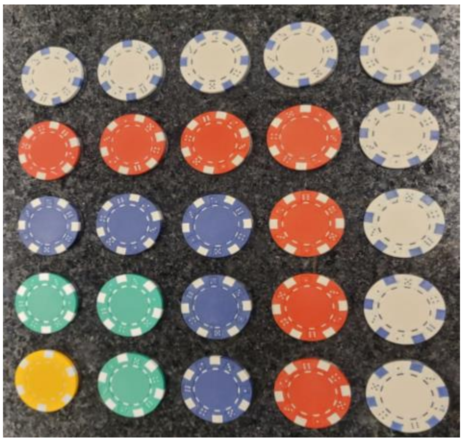
Figure 6: Square numbers created by the students.

Source: student work photographed by the author (2024).

A brief explanation of the historical importance of this sequence was presented: it was based on this sequence that the famous Pythagorean theorem was derived! As clarified by Mendes and Yamamoto (2022), each term of the visual sequence of odd numbers (the L-shaped figure resulting from the subtraction of consecutive squares) was called by the Pythagoreans "gnomon". In the case of the Pythagorean theorem, two square figures added together result in a third. Thus, the square of the smaller number needs to become one or more gnomons that fit into the second one, forming the third element of a Pythagorean triple.

Following Mendes and Yamamoto (2022), discussions were held with the students about the origin of the word "gnomon" and its use in astronomy. Then, we noticed how consecutive gnomons - which are odd numbers - form a square number when joined since the first one. It is a visual way to explain, for example, why 1 + 3 + 5 + 7 + 9 = 25, as illustrated in the following figure.

Figure 7: Gnomons of sizes 1 to 5 forming together the 25-chip square number.



Source: student work photographed by the author (2024).

The teacher and students together generalized the idea until the following identity was obtained:

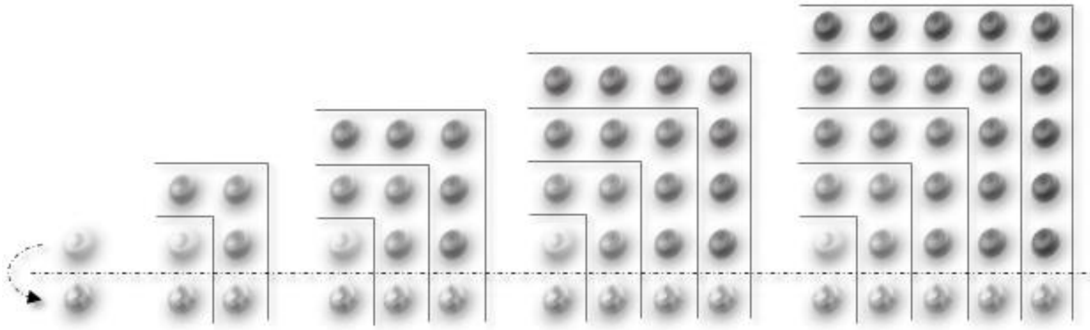
$$1 + 3 + 5 + \cdots + 2n - 1 \quad = n^2$$

This formula expresses that if we add the first n odd numbers, we obtain as a result n^2. The reason for this became evident through the activity performed with the chips, transposing it into algebraic language being a later concern.

To end the class, one last question was provided: if the sum of the first n odd numbers is n^2, then what would be the sum of the first n even numbers? Students were oriented to use a similar visual reasoning to solve the question. As a hint, students were asked to simply slightly modify the odd number figure (Figure 7).

After some groups found the answer, they were asked to try to generalize algebraically as done with the sum of the odd numbers. We then discussed the obtained visual representation (Figure 8) and its historical significance.

Figure 8: Sequence of the first oblong numbers obtained from the square numbers.



Source: developed by the author (2024).

We know that gnomons are odd numbers. Therefore, if we add an extra chip to one of the arms of the gnomon, we will obtain even numbers. It is enough now to add these "even gnomons" together, obtaining as a result the sum of the first n even numbers. By observing the constructions, it is noticeable that they form rectangles, with one side being larger than the other by one unit.

Thus, the sum of the first two even numbers is given by $2 \times 3 = 6$, of the first three, $3 \times 4 = 12$, and so on. The sum of the first five even numbers will be, then:

$$2 + 4 + 6 + 8 + 10 = 5 \times 6 = 30$$

In general, we conclude that the sum of the first n even numbers is:

$$2 + 4 + 6 + \cdots + 2n = n \times (n + 1)$$

This operation is carried out by multiplying the side of the square number by a side with one extra chip.

According to Heath (1921), this type of number was called "oblong". Any rectangular numbers were referred to as "prolate", but when formed by a sequence of even numbers, with one side one unit larger than the other, they are specifically called "oblong".

The antagonism between square and oblong, tied to the idea of even and odd, is depicted within the famous Pythagorean table of opposites as described by Roque (2012). For the doctrine, duality was a factor of utmost importance, primarily represented by the contrast between even and odd, stemming from the cosmogony of the One. The table of opposites is as follows:

Figure 9: Pythagorean table of opposites.

One	Many
Odd	Even

Square	Oblong
Limited	Unlimited
Straight	Curve
Light	Dark
Male	Female
Good	Evil

Source: developed by the author (2024).

In agreement with the detailed explanation in the first lesson about the Pythagorean cosmos, this sect believed that the One would be the origin of the entire Universe, a bisexual being from which everything that exists originated. The One is not a number because the word "number" implies plurality. Therefore, the One is defined as the opposite of plurality - a concept in evident contrast with the current conceptions of numbers known by the students. The odd and the even deal with another aspect of this dichotomy, with the former promptly related to unity and the latter to abundance.

As studied, square numbers are formed by a sequence of odds, and oblongs by a sequence of evens. Thus, "square" aligns with "odd" and "oblong" with "even". "Limited", "light", "straight", and "good" refer to things predictable by the human intellect. Square numbers, for example, always have the same form. Meanwhile, "unlimited" seems to correspond with the denomination of "oblong", which are shapes each with a distinct ratio between the sides (1:2; 2:3; 3:4; 4:5...). "Darkness", "curved", and "evil" similarly refer to undefined or difficult-to-understand concepts.

The presence of the woman on the "negative" side of the table aligns with the influence of women in ancient Greek society, which was minimal and generally distant from intellectual productions. Everything dictated by reason, according to the Pythagorean sect, would be superior to the undefined or relative.

3.3 Visualizing the triangular number formula

The guiding question of this lesson was simple yet significant: how can we quickly determine the number of chips in a triangular number by knowing its side length? For example, how can we determine the number of chips in a triangle with a side length of 107? The students already knew that in the case of a square, all it took was to multiply one side by the other, but this case is not as straightforward. Constructing a triangle with a side length of 107 is also out of the question.

The first problem of the lesson was quite straightforward: it required students to think of a way to perform this quick counting. It was not expected that they would immediately succeed, so hints were offered every five minutes. They were as follows: (i) When we don't know how to solve a problem, it can be a good idea to reduce it to another problem that we know how to solve. Remember what was studied in the previous lesson; (ii) It's difficult to count a triangle directly because we don't have a practical formula for counting. But if we observe the square or the oblong numbers, we can calculate the number of chips by multiplying their sides. Is it possible to use them to count the triangle? (iii) Try combining triangles to form rectangles or squares, and then perform the reckoning from there.

From the second hint, some groups indeed managed to find the method, while one of them required the final hint. The following figure illustrates the necessary reasoning:

Figure 10: Square number (left) and oblong number (right) divided into two triangles.

Source: student work photographed by the author (2024).

Every square number can be divided into two consecutive triangular numbers. Additionally, every oblong number consists of two identical triangular numbers. This information paves the way to find the sum of the chips composing any triangular number.

The conclusion to be reached is that to find the number of objects in a triangular number, one simply needs to find half of the corresponding oblong number. Therefore, if we want the number of objects in the eighth triangular number, we simply multiply 8 by 9 to find the oblong number. Then, we divide it by two, obtaining 36. Applying this to 107, the answer would be 107 multiplied by 108 divided by 2, resulting in 5,778.

It's important to relate this to the formation of triangular numbers. If the triangular number with side length n is formed by $1 + 2 + 3 + 4 + \cdots + n$, and this sum is equal to$n(n + 1)/2$, then we immediately have:

$$1+2+3+4+\cdots+n=\frac{n(n+1)}{2}$$

To quickly reinforce the application of what had been discovered, some exercises and problems were offered to the students:

1. How many chips will the triangle have for each side length: (a) 109? (b) 5003? (c) 80020?
2. Triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91… We can notice that the sum of two consecutive triangular numbers is always a square number. Explain why this happens.
3. Is it true that the sum of eight times a triangular number plus one is always a square number? Explain.

The last two questions again required visual reasoning, for which the chips proved to be quite useful. These properties had been noted by the Pythagoreans since antiquity, as mentioned by Roque (2012) and Heath, and their understanding simply requires the spatial rearrangement of triangles, exactly what the students had been doing since the beginning of this lesson. After reminding the groups of this method, most were able to solve the problems without major difficulties, requiring only a good deal of effort.

3.4 Visualizing the AP formula

In this penultimate class, we took the final steps toward the general formula for arithmetic progressions (AP). Equipped once again with the chips, the students were presented with the following problems:

> We know how to count sequences that start at 1, modeling them as triangular numbers. But how do we proceed if they don't start at 1?

1. Representing as part of a triangular number:

   (a) 3+4+5+6+7+8 (b) 11+12+13+14+15

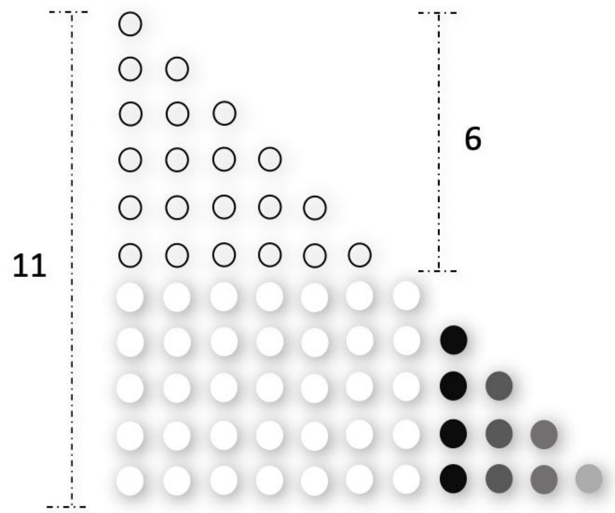
2. Reflecting on the previous figurative constructions, describe a way to quickly calculate the sum of any sequence of consecutive numbers. Write a formula for the calculation. Hint: One possibility is to think of the sequence as part of a figurate triangle.

3. Show that your formula works for the cases presented in (a).

4. Calculate $101 + 102 + 103 + 104 + \cdots + 205 + 206$.

As depicted in Figure 10, one possible solution is to separate the figure into a rectangle (white area) and a triangle (gradient), calculating both separately. Another approach, which is analyzed in this article, would be to consider the way a subtraction of two figurate triangles (in Figure 10, subtracting the triangle with side 6 from the one with side 11) is formed. The students successfully arrived at these answers.

Figure 11: Sum of the sequence from 7 to 11 in a figurative representation - Approach I.



Source: developed by the author (2024).

Let the side of the smaller triangle be l and that of the larger one be L. An expression for such subtraction would be:

$$\frac{L \times (L+1)}{2} - \frac{l \times (l+1)}{2}$$

Simplifying:

$$\frac{L^2 - l^2 + L - l}{2} = \frac{(L+l)(L-l) + (L-l)}{2} = \frac{(L-l)(L+l+1)}{2}$$

Considering the initial term as a_1, the final one as a_n, and the number of terms as n, one can write the equivalences:

$$a_1 = l + 1$$

$$a_n = L$$

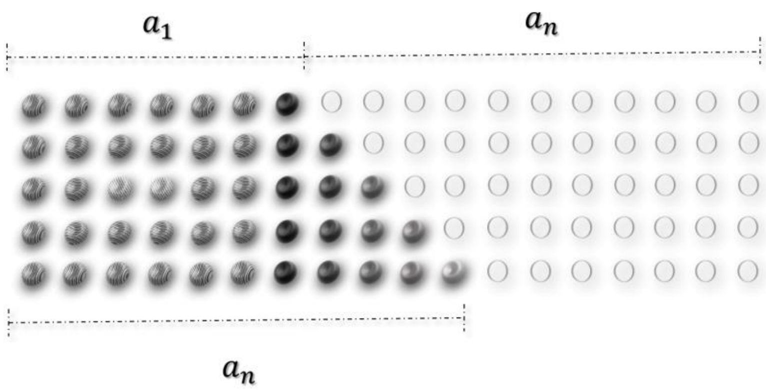
$$n = L - l$$

$$\frac{(L-l)(L+l+1)}{2} = \frac{(a_1 + a_n)}{2} n$$

Thus, we reached alongside the students the most widespread formula for the sum of the terms of an arithmetic progression. However, one question remains: isn't there a simpler way to arrive at this formula? Shouldn't it be possible to visualize it figuratively?

To finish this discussion, we looked again at the shape of the chips: they actually form a trapezoid. According to the formula of the arithmetic progression, if we add a_1 and a_n as a horizontal line and multiply it n times, we obtain a duplication of the trapezoid that fits perfectly into the original figure, forming a rectangle (Figure 11). In other words, to calculate the number of chips in the original figure, we just need to divide this rectangle in two, as done in the formula. Thus, the algebraic representation is justified.

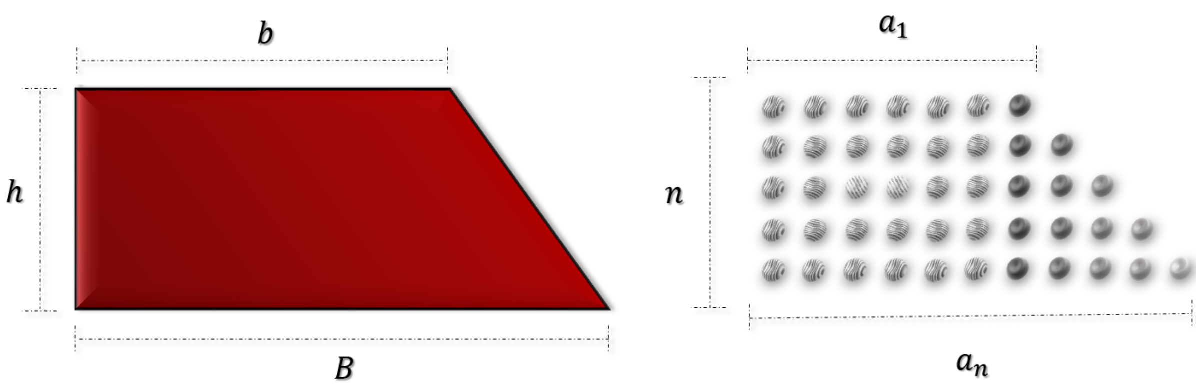
Figure 12: Sum of 7 up to 11 in figurative representation - Approach II.



Source: developed by the author (2024).

To conclude, we looked at the formula again. Its appearance certainly resembled a formula that the students were very familiar with! One of the students remembered: it's the formula for the area of a trapezoid. After all, in this activity, we are calculating the number of chips in a figurative trapezoid, and there lies the origin of such similarity.

Figure 13: Comparison between the trapezoid and the AP.

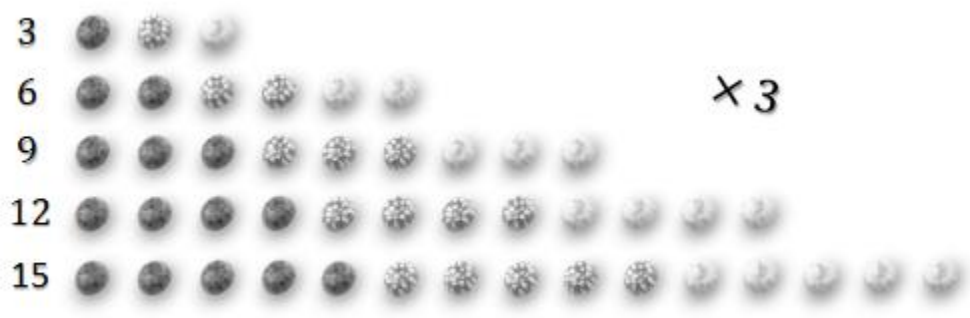


Source: developed by the author (2024).

$$\frac{(b+B)}{2}h = \frac{(a_1+a_n)}{2}n$$

Finally, a valid question within the study of sequences is: what if the ratio is different from 1? Why does the formula still work? For example, for $3+6+9+12+15$, we would figuratively have:

Figure 14: Sum of a sequence with a ratio of 3.



Source: developed by the author (2024).

As the same figurative procedure, in the formula, we would simply be duplicating this figure and fitting it onto itself. Therefore, the formula still successfully calculates the number of chips.

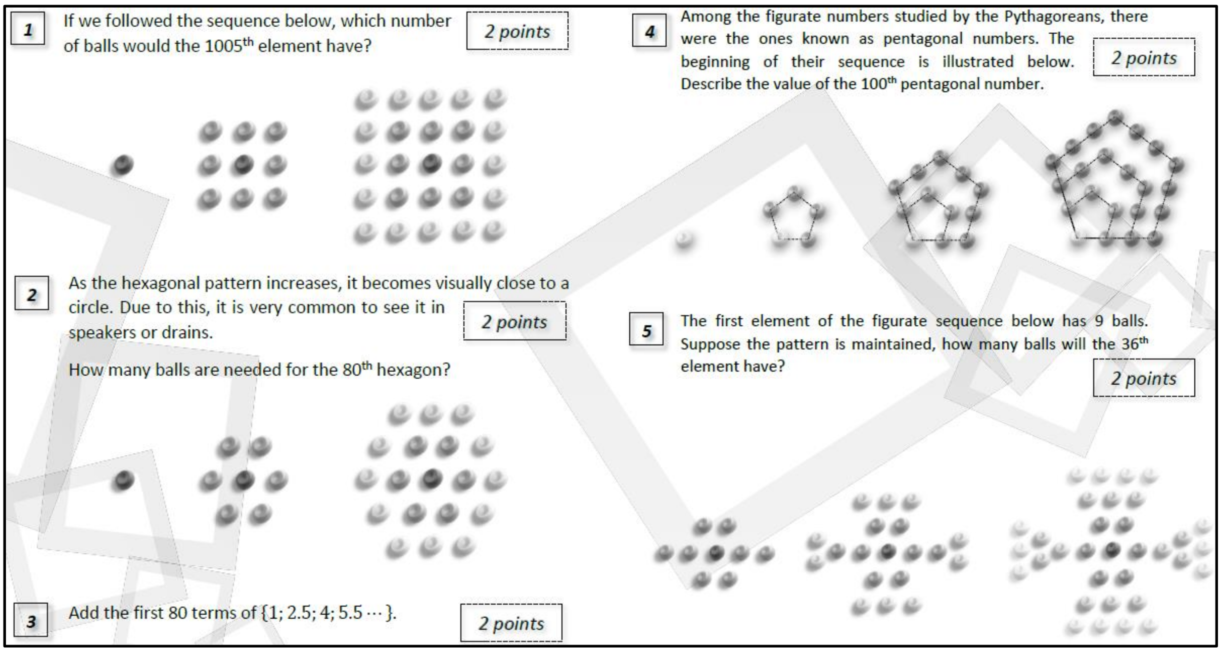
It was emphasized how, historically, there were no algebraic formulas in the time of the Pythagoreans: all study around sequences was done using the figures themselves, with properties visually evident. According to Roque (2012), algebra was an invention of the Arabs many centuries later. The articulated use of various representations is something we are capable of doing today thanks to the joint efforts of various peoples over many millennia.

3.5 Evaluative Problems

In the previous class, the students found and visualized the formula for arithmetic progression (AP). In this last section, additional problems requiring an application of this knowledge were addressed to illustrate the practical significance and the functioning of arithmetic progressions. Student learning was also assessed.

The students were divided into groups and collectively worked on the following set of problems, provided to them on paper:

Figure 15: Evaluative problems.



Source: developed by the author (2024).

The teacher circulated around the classroom to monitor the student's progress on the activity. Students made various sketches on the figures, breaking them down into simpler shapes so they could be counted in a manner similar to what was done previously. Their discussions focused on conjectures about how to properly divide the figures for counting.

At times, some students struggled with the counting process: in problem 2, for instance, the triangular numbers that compose the hexagon only appear from the second term onwards. Thus, the third term will have six triangular numbers of side 2, the fourth will have triangular numbers of side 3, and so on - highlighting the need to subtract one from the term to obtain the side of the triangle, which can be somewhat confusing when performing calculations. However, this type of difficulty is common in this mathematical content, as calculating any term of the AP requires multiplying the ratio by the number of terms minus one, which often leads to recurring confusion for many students.

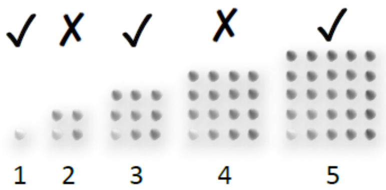
Nevertheless, many of the responses showed notable creativity, as each figurative problem has numerous solutions. Below is a record of some of the obtained responses.

Figure 16: Some obtained answers.

Answers

1) Through the square number sequence:

It's the sequence of odd sided square numbers. We can obtain it by removing the even sided squares from the square number sequence.

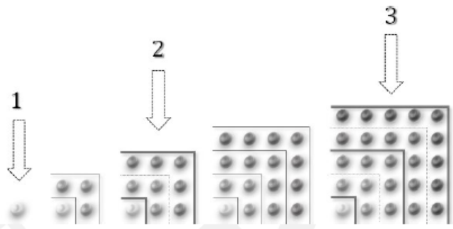


To get to other odd numbers, we must skip one square each time starting from 1. Therefore, to get to the 1005th odd sided square, we need to make $1 + 1004 \times 2$. That is, we advance 1004 steps of 2 in the square number sequence. The odd number obtained is 2009.

The answer is, therefore, $2009^2 = 4\ 036\ 081$.

By concatenating odd numbers:

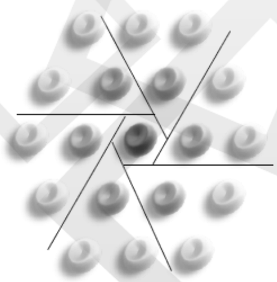
If we relocate the initial dot in the leftmost lower part, we can see 2 gnomons are added in each step. Hence, the 1005th term received the addition of 1004×2 odd numbers to the initial dot. Thus, it will be the square of side 2009, with 2009^2 balls.



2) Centered hexagonal numbers grow according to the triangular numbers that compose them. Hexagon number 80 will have 6 triangular numbers of side 79 clinging to the central dot.

The number of dots in a n-sided triangular number can be reckoned as $\frac{n(n+1)}{2}$, hence the total in the 80th figure is: $\frac{79 \times 80}{2} \times 6 + 1$.

It results 18961.



3) The 80th term is $1 + 79 \times 1{,}5 = 119{,}5$, which makes the sum of the 80 first terms be $\frac{(1+119{,}5)80}{2} = 4820$.

Source: developed by the author (2024).

Most groups scored above 8 on this activity, with one group getting all the questions right and another scoring only 5 points. It was concluded that the students were able to develop good pictorial reasoning and an effective understanding of arithmetic progressions.

4 Conclusion

The students demonstrated considerable enthusiasm for working with manipulative materials, as it is an opportunity that is virtually absent in high school mathematics. Furthermore, they were able to solve the problems within the expected time frame, indicating that the difficulty progression was appropriate.

The discussion on the history of mathematics engaged many students with an affinity for humanities. By sharing their memories and reasoning, the students managed to reach a consensus, expanding their understanding of ancient societies. At the end of the class, one student expressed surprise at the profound impact of mathematics on society, considering the discussion highly beneficial.

During the mathematical activity, some students showed anxiety about finding a "magic formula." Many were accustomed to a learning approach based on memorization and formula application, this kind of interactive investigation being a novelty for them. To address this, it was important to clarify the objectives of each stage of the process.

The connection between different areas of mathematics was another aspect satisfactorily achieved. Students emphasized the importance of visualizing complex concepts, something they had not experienced in years, as mathematics previously consisted mainly of algebraic formulas. This approach, recommended by the BNCC (2018), was well received by students, who could now understand formulas in a more tangible way.

This article also aimed to present high school teachers with a more historically integrated approach to arithmetic progression, avoiding elitist anecdotes. The connection between arithmetic progression and the trapezoid formula is not widely known, but this figurative representation aimed to make this correlation more tangible. It is hoped that this experience will inspire other teachers to adopt approaches richer in connections, allowing students to understand the importance of mathematics in society and the diversity of mathematical thinking.

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